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藤家龍雄

主論文

Boundary behaviors of Dirichlet functions

by

Tatsuo Fuji'i'e

(Received September 12, 1969)

Introduction

Since the ideal boundary of a Riemann surface was introduced by R. S. Martin, H. L. Royden, Z. Kuramochi, S. Mori, Y. Kusunoki and C.Constantinescu - A.Cornea, the theory of Riemann surfaces has been studied by plenty of methods in intuitive aspects. Various theorems on functions in plane regions were extended to the case of functions on Riemann surfaces. And the success in these extensions is due to introduction of harmonic measure and capacity, etc. on the ideal boundary analogous to those on boundaries of plane regions.

We shall study, in this paper, Dirichlet functions (BLD functions) on Riemann surface. Kuramochi boundary contributes so many tools such as potentials, capacity and fine topology for the study of functions of this kind. Also, the extremal length plays an effective role for Kuramochi compactification.

Main purpose of this paper is to extend Beurling's two theorems (§3, §5) of Fatou type and of Riesz type. One of the Kuramochi's original purpose of introducing his boundary was in the extension of Beurling's theorems.

In the extensions of the theorem of Fatou type by Kuramochi [13] [14] and Constantinescu-Cornea [4], Kuramochi boundary and Kuramochi capacity were used instead of the unit circle and logarithmic capacity in Beurling's theorem, and a filter at a minimal point of Kuramochi boundary instead of angular domains. And instead of a meromorphic function in the unit disk whose Riemannian image has finite spherical area, Constantinescu-Cornea and Kuramochi used, respectively, a Dirichlet mapping and an analytic mapping generating an almost finitely sheeted covering surface of the image surface. The image surface is compactified in their own ways.

To attempt a further extension we shall introduce two notions: a GD mapping making use of an analytic mapping and compactification satisfying the separability axiom with respect to extremal distance for the image surface (See §3) .

For the theorem of Riesz type, Kuramochi [17] extended it by introducing the notion of ordinary values in the sense of Beurling for a set of inner points of the image surface. However, for boundary points it loses the meanings. We need use of Evans potential (§4) to restore it.

Analytic mappings of type Bl (§6) have interesting behaviors in relation with the theorem of Riesz type.

This paper consists of the following six sections.

§ 1 Kuramochi boundary and Extremal length (Preliminaries)

1 Kuramochi boundary Kuramochi compactification of a Riemann surface plays an important role in the study of Dirichlet functions (functions of Beppo Levi-Deny). Points of Kuramochi boundary are thought, in a sense, to be inner points, and the notion of capacity is defined for sets of the boundary points.

Before introducing Kuramochi compactification we refer to Dirichlet principle on a Riemann surface R. Let f be a Dirichlet function and F be a non-polar closed set. There exists a unique Dirichlet function f^F(the solution of Dirichlet principle), which is harmonic in R - F and equals to f on F, such that

$$D(f^F) = \min \left\{ D(f'); \ f' - f \in D^F \right\},$$

where D^F denotes the class of Dirichlet functions which are zero quasi-everywhere on F. If G is a region for which $R - G$ is not polar, the map $f \to f^{R-G}(a)$, $a \in G$, is a linear and positive functional on $C_0^\infty(R)$. So, there exists a measure μ_a^G on the boundary of G such that for every $f \in C_0^\infty(R)$ $f^{R-G}(a) = \int f d\mu_a^G$. If f is a Dirichlet function and if G has compact boundary, then f is μ_a^G-summable and

$$f^{R-G}(a) = \int f d\mu_a^G .$$

We fix a compact disk K_0 in R and put $R_0 = R - K_0$. A superharmonic functions in R_0, continued as zero on K_0, is called full-superharmonic if it satisfies the condition:

$$s(a) \geq \int s \, d\mu_a^G \qquad \text{for all } a \in G,$$

for all regions G with compact boundary such that $K_0 \cap \bar{G} = \emptyset$.

Kuramochi compactification R_N^* of a Riemann surface R is given as follows. Let K_0 be a compact domain such that $R_0 = R - K_0$ is connected. The function $N(z,p)$, called N-Green function, is determined by the following conditions:

1) $N(z,p)$ is harmonic in R_0 except at p.
2) $N(z,p)$ has a logarithmic singularity at p with the coefficient 1.
3) $\lim_{z \to K_0} N(z,p) = 0$.
4) For a compact set K in R which contains K_0 and p in its interior, $N(z,p)$ is the solution of Dirichlet principle; $N^K(z,p) = N(z,p)$.

Now let $\{p_i\}$ be a sequence of points in R_0 which does not cluster in R_0. When z stays in any compact domain, $\{N(z,p_i)\}$ is, from some i on , a uniformly bounded sequence of harmonic functions of z hence forms a normal family. A sequence $\{p_i\}$, for which the corresponding $\{N(z,p_i)\}$ converges to a harmonic function, is called fundamental. Two fundamental sequences are called equivalent if the corresponding $N(z,p)$'s have the same limit. The class of all fundamental sequences equivalent to a given one determines an ideal boundary point of R. The set of all such ideal boundary points are denoted by Δ_N , and $R_N^* = R \cup \Delta_N$. R_N^* is independent of the choice of K_0 and the pole of $N(z,p)$. In the sequel, we take as K_0 a parametric disk of R.

Equivalent definition of Kuramochi compactification was given by C.Constantinescu and A.Cornea. Namely, let N denote a class of continuous Dirichlet functions f for which there exists a compact set K_f such that $f = f^{K_f}$. R_N^* is the compactification of R such that all functions of N have continuous continuation on the boundary Δ_N and points of Δ_N are separated by the extended functions. R_N^* is metrizable.

Points of Kuramochi boundary Δ_N are divided into two sets Δ_1 and Δ_0 . A point p of Δ_1 is called <u>minimal</u> and distinguished from points of Δ_0 by the condition that $N(z,p)$ can not be represented by a sum of non proportional potentials of the form $\int N(z,q)\, d\mu$. This definition of minimal points coincides with that of Kuramochi (cf. [14] [28] [34]).

2 Extremal length Various definitions of the extremal length have been considered since it was first introduced by Ahlfors and Beurling.

For our purpose we use the following definition : let Γ be a class of locally rectifiable curves c[1], and Φ be the class of non negative covariants ρ which satisfy $\iint_R \rho^2\, dxdy \leqq 1$ and for which $\int_c \rho |dz|$ are determined ($\leqq \infty$) for every $c \in \Gamma$. The extremal length λ_Γ of Γ is defined as

$$\lambda_\Gamma = \left(\sup_{\rho\in\Phi} \inf_{c\in\Gamma} \int_c \rho |dz| \right)^2$$

and each ρ of Φ is called <u>admissible</u> for the problem of the extremal length.

The extremal distance $\lambda(E_1,E_2)$ between two sets E_1 and E_2 is the extremal length of the family Γ of curves which join E_1 and E_2. If either E_1 or E_2 is void $\lambda(E_1,E_2)$ is defined as infinity. In the case that both E_1 and E_2 are closed, we may take, as a curves of Γ, a connected arcs in $R - E_1 \cup E_2$.

We prove the following

Theorem 1.1. ([8],[21],[27])[2] Every curve which starts from a point of R and tends to the ideal boundary converges to a point of the Kuramochi boundary except for curves belonging to a family of infinite extremal length.

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1) In this paper all curves are assumed to be locally rectifiable so far as we deal with the extremal length. A curve in an open subset of R is said to tend to the ideal boundary if, for any compact subset K, an end part of the curve is disjoint from K.

2) This theorem will be extended in § 3 .

Proof Since the Dirichlet integral $D(N(z,p))$ over R_0 outside a neighborhood V of the pole p is finite, $\rho = |\text{grad } N(z,p)| / \sqrt{D(N(z,p))}$ is admissible for the problem of extremal length of a family of curves in $R_0 - V$. Let Σ denote the family of curves which start from inner points of R and tend to the ideal boundary, and Σ_0 denote the sub-family of Σ each curve of which does not converge to any point of Kuramochi boundary. By definition $N(z,p) = N(p,z)$ for z and p in R_0, and along any curve of Σ_0 $N(p,z)$ has not a limit. So, every curve of Σ_0 has infinite length by the metric

$$\rho'|dz| = \begin{cases} \rho|dz| & \text{in } R - K_0 - V \\ 0 & \text{in } K_0 \cup V \end{cases} .$$

This means that Σ_0 has the infinite extremal length.

3 Kuramochi capacity Let E be a compact set on $R_0^* = R_N^* - K_0$ and E_n be the $1/n$ -neighborhood of E.

Let $\{R_m\}$ be a regular exhaustion of R and ω_{nm} be a function in R_m such that ω_{nm} is harmonic in $R_m - \bar{E}_n - K_0$,

$$\omega_{nm} = \begin{cases} 0 & \text{on } K_0 \\ 1 & \text{on } \bar{E}_n \cap R_m \end{cases}$$

and the normal derivative $\frac{\partial \omega_{nm}}{\partial \nu} = 0$ on $\partial R_m - \bar{E}_n$. The Dirichlet integral $D(\omega_{nm})$ over R_m is equal to the reciprocal of the extremal distance λ_{nm} between K_0 and $\bar{E}_n \cap R_m$ measured in R_m. The sequence $\{D(\omega_{nm})\}_m$ is monotone increasing with m and if it has a finite limit, $\{\omega_{nm}\}_m$ converges locally uniformly to ω_n in $R - \bar{E}_n - K_0$. And

$D(\omega_{nm})$ and λ_{nm} tend to the Dirichlet integral $D(\omega_n)$ over R and the extremal distance between K_o and $\overline{E}_n \cap R$, respectively.

The sequence $\{\omega_n\}$ converges in Dirichlet norm, and hence converges uniformly on every compact set on R_o. Consequently, the limit $\omega_E = \omega(z,E)$ is harmonic in $R_o - E$ and has the finite Dirichlet integral $D(\omega_E)$ over R. Kuramochi capacity $C(E)$ of E is defined as $C(E) = D(\omega_E)$ and ω_E is called the capacitary potential of E.

For the latter use, we refer to the potential theory on R_N^* developed by Constantinescu and Cornea. (For details see [4])

As they showed, the continued N-Green function $N(p,q)$ on $R_o^* = R_N^* - K_o$ is lower semi-continuous on $R_o^* \times R_o^*$ and symmetric; $N(p,q) = N(q,p)$. We consicder the potential

$$p^{\mu}(p) = \int N(p,q)\, d\mu(q), \quad (\mu(R_o^*) < \infty)$$

generated by a measure μ on R_o^*. If $\mu(\Delta_o) = 0$ μ is called canonical . $\|\mu\|^2 = \int p^{\mu} d\mu$ is called energy of μ . μ has finite energy if and only if p^{μ} (continued with 0 on K_o) is a Dirichlet function on R, and then

$$\|\mu\|^2 = D(p^{\mu}).$$

We continue a non negative full-superharmonic function s on Δ as

$$s(a) = \sup_{K \subset R_o} s_K(a) \qquad a \in \Delta,$$

where s_K is the infimum of all non negative full-superharmonic functions which are not smaller than s quasi-everywhere on a compact set K in R_o. The s_K belongs to class N and has continuous continuation on Δ. Thus extended s satisfies

$$s(a) = \lim_{R_0 \ni b \to a} s(b) , \quad a \in \Delta_1 .$$

<u>Domination principle</u> Let μ be a canonical measure for which p^{μ} is finite almost everywhere (μ) on R_0^*. If non negative full-superharmonic function s (continued on Δ) majorates p^{μ} almost everywhere (μ), then $s \geqq p^{\mu}$ on R_0^*.

<u>Kuramochi capacity</u> C(E) of a compact set E is defined as

$$C(E) = \sup_{\mu} \mu(E),$$

where μ runs over all canonical measure for which $p^{\mu} \leqq 1$.

<u>Equilibrium principle</u> For a compact set E in R_0^* there exists a unique canonical measure κ on E such that $p^{\kappa} \leqq 1$, $p^{\kappa} = 1$ on E except for an F_{σ} set of capacity 0 and

$$C(E) = \kappa(E) = \|\kappa\|^2 \overset{\frac{1}{2\pi}}{=} D_{\Delta}(p^{\kappa})^2.$$

We show that this capacity is equal to that of Kuramochi defined before. Let E_n be the 1/n -neighborhood of E such that $\bar{E}_n \cap K_0 = \emptyset$. Then,

$$1_{\bar{E}_n \cap R} = \omega(z, E_n) \quad \text{in } R_0 .$$

On the other hand, there exists a measure μ_n such that

$$1_{\bar{E}_n \cap R} = p^{\mu_n}.$$

And p^{μ_n} converges to the equilibrium potential p^{κ} of E in Dirichlet norm. So we have

$$\|dp^{\kappa} - d\omega_E\| \leqq \|dp^{\kappa} - dp^{\mu_n}\| + \|dp^{\mu_n} - d\omega_n\| + \|d\omega_n - d\omega_E\| \longrightarrow 0$$

for $n \longrightarrow \infty$.

And since $p^{\kappa} = \omega_E = 0$ on K_o, $p^{\kappa} = \omega_E$ on R_o. Consequently,

$$2\pi C(E) = 2\pi\, \kappa(E) = D(dp^{\kappa}) = D(\omega_E).$$

The capacity $C(G)$ of an open set G is defined as $C(G) = \sup_K C(K)$, where K runs over all compact sets in G. There exists a unique equilibrium potential p^{κ} with κ on $\overline{G}$ such that

$$p^{\kappa} \leqq 1, \quad p^{\kappa} = 1 \quad \text{on } G - \Delta_o$$

and $C(G) = \kappa(\overline{G}) = \|\kappa\|^2$. p^{κ} is obtained as the infimum of non negative full-superharmonic functions which are not smaller than 1 on $G \cap R_o$. This shows

$$C(G) = C(G \cap R_o).$$

Outer capacity $C_o(A)$ and inner capacity $C_i(A)$ of a set A are defined in a usual way:

$$C_o(A) = \inf_{G \supset A} C(G), \qquad C_i(A) = \sup_{K \subset A} C(K),$$

where G runs over all open sets containing A, and K runs over all compact sets contained in A.

Using this capacity, the notion of quasi continuity is introduced. A mapping f of R_o^* into a topological space is said <u>R_o^*-quasi-continuous</u> if, for all $\varepsilon > 0$, there exists an open set G in R_o^* such that $C(G) < \varepsilon$ and the restriction of f on $R_o^* - G$ is continuous. If a mapping f of R_N^* into a topological space is R_o^*-quasi-continuous in R_o^* for all K_o, it is said <u>quasi-continuous in</u> R_N^*.

A set $M \subset R_N^*$ is said <u>full-polar</u> if, for any disk K_o, $M \cap R_o^*$ is of outer Kuramochi capacity 0 with respect to R_o^*.

4 Fine topology   A set M is said thin at a point $a \in R_o^*$ if there exists a full-polar set M' and a non negative full-superharmonic function s such that either $a \notin \overline{M - M'}$ or $a \in \overline{M - M'}$ and

$$\varliminf_{\substack{b \to a \\ b \in M-M'}} s(b) > s(a).$$

R_o is not thin at any point of Δ_1, but R_o^* is thin at all point of Δ_o. For a point $a \in R_o^* - \Delta_o$, a set M is thin at a if and only if $N_M(p,a) \neq N(p,a)$. If F is a closed set in R_o and $a \in R_o^* - \Delta_o$, there exists at most one connected component of $R_o - F$ on which $N_F(z,a) = N(z,a)$. Especially, for only one component V_n of the neighborhood $U_n(a)$ of a, $R_o - V_n$ is thin at a. By this fact we know that every point of Δ_1 is accessible. For any point a of R_o^*, the class of sets on R_o^* which contain a and whose complement is thin at a forms a filter. This filter being independent of choice of K_o, it can be constructed at all points of R_N^* and introduce a finer topology on R_N^*, which is called fine topology. A mapping of R_N^* into a topological space is said fine continuous if it is continuous with respcet to fine topology.

Constantinescu and Cornea proved that a quasi-continuous mapping of R_N^* into a topological space is fine continuous quasi-everywhere[3] on R_N^*.

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3) In the case of R_o^* " quasi-everywhere " means that the exceptional set is full-polar.

§ 2 Continuation of Dirichlet functions on Kuramochi boundary

Several ways of continuation of a Dirichlet function onto the boundary were studied by M. Brelot, J. Deny-J.L.Lions, J. L. Doob, C.Constantinescu - A.Cornea and M. Ohtsuka. In this section we consider Dirichlet functions on a Riemann surface R and deal with its quasi-continuous continuation on Kuramochi boundary and continuation along curves. As already refered, the quasi-continuous continuation of a Dirichlet function on R_N^* is fine continuous quasi-everywhere.

1. For the quasi-continuous continuation Constantinescu-Cornea [4] proved the following Lemma and Theorems 2.1. and 2.2.

Lemma 2.1. (cf. also [6]) Let f be a real quasi-continuous function on R_N^* which is zero on K_o and whose restriction in R is a Dirichlet function. Then, for any $\alpha > 0$, the outer Kuramochi capacity

$$c_o(\{a \in R_o^* \ ; \ |f(a)| \geqq \alpha\}) \leqq \frac{\|df\|^2}{2\pi\alpha^2} .$$

Theorem 2.1. A Dirichlet function in R has a quasi-continuous continuation on R_N^*.

Following Constantinescu-Cornea, we give the proof as we need later some notions in the proof.

Proof Let f be a Dirichlet function on R and $\{K_n\}$ be an exhaustion of R such that $K_o \subset K_1$ and

$$\|df^{K_n} - df^{K_{n+1}}\| < \frac{1}{3^n} .$$

The f^{K_n} has continuous continuation on R_N^*, and we denote by f^{K_n} again the continued function. By above Lemma we have

$$C_o(\{a\in R_o^*;\ |f^{K_n}(a) - f^{K_{n+1}}(a)| \geqq \frac{1}{2^n}\}) < \frac{1}{2\pi}(\frac{2}{3})^{2n}.$$

Hence there exists an open set $G_n \subset R_o^* - K_{n-1}$ such that

$$C(G_n) < \frac{1}{2\pi}(\frac{2}{3})^{2n}$$

and

$$|f^{K_n} - f^{K_{n+1}}| < \frac{1}{2^n}$$

on $R_o^* - G_n$. This means that the series

$$f^{K_k} + \sum_{n=1}^{\infty}(f^{K_{n+1}} - f^{K_n})$$

is uniformly convergent on $R_o^* - \bigcup_{n=m}^{\infty} G_n$ and equals f on R. This series converges outside the set $\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} G_n \subset \Delta$. We continue f as

$$f = \begin{cases} 0 & \text{on } \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} G_n \\ f^{K_1} + \sum_{n=1}^{\infty}(f^{K_{n+1}} - f^{K_n}) & \text{on } \Delta - \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} G_n. \end{cases}$$

For any $\varepsilon > 0$, there exists a number m such that

$$C(\bigcup_{n=m}^{\infty} G_n) < \frac{\varepsilon}{2}.$$

Since the restriction of f^{K_n} on R_o is a Dirichlet function there is an open set $G_n' \subset R_o$ such that $C(G_n) < \varepsilon/2^{n+1}$ 4) and the restriction of f^{K_n} on $R_o^* - G_n$ is continuous. Hence the restriction of f on

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4) Let $G(z,p)$ be the Green function in R_o with the pole at p, then $G(z,p) \leqq N(z,p)$.. So we have
($N(z,p)$-capacity of G_n) $\leqq$ ($G(z,p)$-capacity of G_n) $< \varepsilon/2^{n+1}$.

$R_0^* - \bigcup_{n=m}^{\infty} (G_n \cup G_n')$ is continuous. And since the capacity

$$C\Big(\bigcup_{n=m}^{\infty} (G_n \cup G_n')\Big) \leqq C\Big(\bigcup_{n=m}^{\infty} G_n\Big) + C\Big(\bigcup_{n=m}^{\infty} G_n'\Big) < \varepsilon ,$$

the proof of the theorem is completed.

As a consequence of the theorem,

Theorem 2.2.(H.Cartan) A full-superharmonic function is quasi-continuous on R_0^*.

2. Continuation of a Dirichlet function along Green lines was first studied by M. Godefroid[11]. According to his result, a Dirichlet function has limits along almost all (Green measure) Green lines. Ohtsuka extended this result as follows, by using B. Fuglede's theorem[7]: every Dirichlet function is absolutely continuous along every curve except for curves belonging to a family of infinite extremal length.

Theorem 2.3. (Ohtsuka[27]) Every Dirichlet function f on R has a finite limit along every curve except for curves belonging to a family of infinite extremal length.

We give a short proof. If f has not limit along a curve c

$$\int_c |\operatorname{grad} f|\,|dz| \geqq \int_c |df| = \infty .$$

We denote by Γ the family of such curves as c, and put

$D(f) = \iint_R |\operatorname{grad} f|^2\, dxdy$. Then $\rho = |\operatorname{grad} f| / \sqrt{D(f)}$ is admissible for λ_Γ, and $\lambda_\Gamma \geqq (\inf_\Gamma \int \rho |dz|)^2 = \infty$.

Since a set of Green lines of vanishing Green measure has the infinite extremal length, Ohtsuka's theorem contains Godefroid's one.

We recall that every curve tending to the ideal boundary of R converges to a point of Kuramochi boundary.

Theorem 2.4. Let $\{f_n\}$ be a sequence of Dirichlet functions converging to a Dirichlet function f in Dirichlet norm such that $f = f_n$ on K_0 for all n, and Γ be the family of all curves which start from points of K_0 and tend to the ideal boundary. Then, there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that the limit $\ell_{n_i} = \lim_c f_{n_i}$ along a curve c of Γ converges to $\ell = \lim_c f$ for $n_i \to \infty$, for every curve c of Γ except for curves belonging to a family of the infinite extremal length.

Proof We take the subsequence $\{f_{n_i}\}$ as $\|df - df_{n_i}\| < \frac{1}{3^n}$, and denote it by $\{f_n\}$ again. The f has limit along every curve of Γ except for curves belonging to a family Γ_f of the infinite extremal length, and so does each f_n along every curve of Γ except for curves belonging to each family Γ_n of the infinite extremal length, respectively. The union Γ' of Γ_f and all Γ_n has the infinite extremal length. Let $E_n = \left\{ z \in R - K \; ; \; |f(z) - f_n(z)| \geq \frac{1}{2^n} \right\}$, where K is a compact set in R containing K_0, and Γ_n^0 be the subfamily of Γ each curve of which intersects E_n. Then, since

$$\rho = |\operatorname{grad}(f - f_n)| \Big/ \|df - df_n\|$$

is admissible for the problem of the extremal length λ_n of Γ_n^0, we have

$$\lambda_n \geq \left(\int_c \rho |dz| \right)^2 \geq \frac{1}{2^{2n}} \Big/ \|df - df_n\|^2 = \left(\frac{3}{2}\right)^{2n} .$$

The extremal length λ ; $\lambda^{-1} = \sum_{n=m}^{\infty} \lambda_n^{-1}$, of the family of curves which intersect $\bigcup_{n=m}^{\infty} E_n$ will be arbitrarily small for sufficiently large m. And if, along a curve c of $\Gamma - \Gamma'$, $\ell_n = \lim_c f_n$ does not converge to $\ell = \lim_c f$, c intersects $\bigcup_{n=m}^{\infty} E_n$ for every m. Hence the extremal length of the family of such curves as c is infinite. This complete the proof.

Doob [5] proved a theorem : Let f_n and f be BLD functions on R. if $f_n \longrightarrow f$ in BLD sense

$$\lim_{n \to \infty} \int |f_n'(\gamma) - f'(\gamma)|^2 \, d\gamma = 0,$$

where $f_n'(\gamma)$ and $f'(\gamma)$ denote the limits of f_n and f along Green line γ , respectively, and $d\gamma$ denotes Green measure. For a suitable supsequence as above, this theorem is a consequence of our theorem.

Now, we compare the quasi-continuous continuation of a Dirichlet function with its continuation along curves. We take, as in the proof of Theorem 2.1, an exhaustion $\{K_n\}$ of R for which $\| df^{K_n} - df^{K_{n-1}} \| < \frac{1}{3^n}$. The f^{K_n} is continuous on R_N^* outside K_n, and the boundary values converge to those of continued f at points of $\Delta - \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} G_n$. The extremal length of the family Γ_0 of curves joining K_0 and the set $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} G_n$ is infinite. In fact, for open sets $O' = \bigcap_{m=1}^{N} \bigcup_{n=m}^{\infty} G_n$ and $O = R \cap O'$, $C(O') = C(O)$. There exists an increasing sequence $\{F_n\}$ of compact sets in R such that $C(F_n) \uparrow C(O)$ and any point of O is contained in the interior of some F_n. The family Σ of curves joining

K_o and O is the union of the family Σ_n of curves joining K_o and F_n. Put $\Sigma'_m = \bigcup_{n=1}^{m} \Sigma_n$, then $\Sigma'_m \uparrow \Sigma$ and, by Suita's lemma [33],

$\lambda_{\Sigma} = \lim_{m \to \infty} \lambda_{\Sigma'_m}$. On the other hand,

$$\lambda_{\Sigma'_m} \geq C(\bigcup_{n=1}^{m} K_n)^{-1} \downarrow C(O)^{-1}.$$

So, we have

$$\lambda_{\Sigma} \geq C(O)^{-1}.$$

Since $C(O) = C(O')$ is arbitrarily small if N is sufficiently large, we conclude $\lambda_{\Gamma_o} = \infty$.

The family of all curves of Γ each of which converges to a point of Kuramochi boundary is denoted by Γ again. Then along every curve of $\Gamma - \Gamma_o$ f has a limit except for curves of a family of the infinite extremal length. If a curve c of $\Gamma - \Gamma_o$ converges to a point p of Δ and if f has a limit f(c) along c, the boundary value $f^{K_n}(p)$ tends to f(c) and to the boundary value of continued f at p because f^{K_n} is continuous on $R^*_N - K_n$. Thus we obtain

Theorem 2.5. The boundary value of quasi-continuously continued Dirichlet function on Δ coincides with its limits along curves, which converge to points of Δ, except for curves belonging to a family of the infinite extremal length.

3. We consider a harmonic Dirichlet function u on hyperbolic R, that is, $u \in HD$. According to Theorem 2.3, u has a limit along every curve tending to the ideal boundary of R except for curves belonging to a family of the infinite extremal length. Suppose the limits of u along curves are constantly zero, hence, of course, the

limits along Green lines are zero. While, according to Brelot's decomposition theorem[2], a Dirichlet function is the sum of a uniquely determined HD-function and an orthogonal function[5], that is, the function whose limits along almost all (Green measure) Green lines are zero. Hence we know our HD-function u is constant zero. Thus we have

Theorem 2.6. (cf.[29]) If an HD-function has vanishing limits along all curves tending to the ideal boundary except for curves belonging to a family of the infinite extremal length, then it is constant zero.

§ 3 Extension of Beurling's theorem of Fatou type

Concerning boundary limits of meromorphic function in the unit disk A, Beurling proved the following theorem.

Theorem 3.1. (Beurling [1]) Let f(z) be a meromorphic function in the unit disk $|z|< 1$ for which

$$\iint_{|z|<1} \left(\frac{|f'|}{(1+|f|^2)} \right)^2 dxdy$$

is finite. Then, f(z) has radial limits on $|z| = 1$ except for a set of vanishing outer capacity.

We call this result Beurling's theorem of Fatou type and extend it to the case of an analytic mapping of a Riemann surface.

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5) It is called Dirichlet potential by Constantinescu and Cornea.

M. Tsuji's extension is to have replaced the radial limits in this theorem by angular limits (Tsuji [35]).

Kuramochi first extended the theorem to the case of an analytic mapping of a Riemann surface. We state his extension.

Let f be an analytic mapping of a Riemann surface R into a Riemann surface R'. The covering surface over R' generated by f is called almost finitely sheeted if f satisfies the following condition: outside a sufficiently large compact set of R', every point is covered a finite number ($\leq M$) of times by the covering surface, and, for every point p of R', there exists a parametric disk K of p such that the part of the covering surface lying over K has a finite total area with respect to the local parameter.

Let R'^* be a metrizable compactification. Let $\{R'_n\}$ be an exhaustion of R' and put $B_n = R' - R'_n$. We denote by $C_p(r)$ a disk centered at $p \in R'^*$ with the radius r. For $C_p(r)$ and fixed parametric disk K'_o in R', we take a Dirichlet function f in R' such that

$$f = \begin{cases} 0 & \text{on } \overline{K}'_o \\ 1 & \text{on } \overline{C_p(r)} \cap B_n, \end{cases}$$

and put $\omega_n = \omega(C_p(r) \cap B_n) = f^{\overline{C_p(r)} \cap B_n}$ and $\omega(C_p(r) \cap B) = \lim_{n \to \infty} \omega_n$, where closures are taken in R'.

R'^* is called HD-separative if the following condition is satisfied for any point p of R'^* and radii r_1 and r_2 such that $r_1 < r_2$, $\overline{C_p(r_i)} \cap K'_o = \emptyset$ ($i = 1,2$) and $\omega(C_p(r_1) \cap B) > 0$: there exists an increasing sequence $\{V_n\}$ of domains with regular relative boundaries such that $\omega(C_p(r_1) \cap V_n) > 0$ and $\omega(C_p(r_1) \cap (R' - V_n))$ decreases to 0 for $n \to \infty$, and a sequence $\{u_n\}$ of Dirichlet functions such that

$$u_n = \begin{cases} 1 & \text{on } \overline{C_p(r_1) \cap V_n} \\ 0 & \text{on } R' - C_p(r_2) \cap R'. \end{cases}$$

Let R_N^* be Kuramochi compactification of R. To each minimal boundary point p, we associate a class $\mathcal{G}_p$ of open sets G of R such that $R_o - G$ is thin at p. The class $\mathcal{G}_p$ forms a filter base and, for every neighborhood U of p, there exists one and only one connected component of $U \cap R_o$ which belongs to $\mathcal{G}_p$. For the filter base $\mathcal{G}_p$, we define the cluster set $f^{\vee}(p)$ of f as

$$f^{\vee}(p) = \bigcap_{G \in \mathcal{G}_p} \overline{f(G)} \quad ,$$

where the closure is taken in R'^*. $f^{\vee}(p)$ is a non-void closed set in R'^*. We call it a <u>fine cluster set</u> of f at p, and when $f^{\vee}(p)$ is one point we call it a <u>fine limit</u>.

Kuramochi's extension of Beurling's theorem is as follows.

Theorem 3.3. (Kuramochi [15] [16]) Let f be an analytic mapping of a Riemann surface R into a Riemann surface R', and R'^* be a metrizable HD-separative compactification of R'. If the covering surface over R' generated by f is almost finitely sheeted, f has a fine limit in R'^* at every point of Kuramochi boundary of R except for points belonging to a set of vanishing inner Kuramochi capacity.

Constantinescu and Cornea treated this problem quite axiomatically. In their theory, R^* is Kuramochi compactification of R and R'^* is a metrizable compactification of R' which is a quotient space of Royden compactification R'^*_D of R'. <u>Dirichlet mapping</u> f(z) of R into R' is an analytic mapping which is continuable to a continuous mapping of R_D^* into R'^*_D.

As an extension of Beurling's theorem,

Theorem 3.4. (Constantinescu and Cornea [4]) Every Dirichlet mapping f has a fine limit in a metrizable quotient space of R'^*_D at every point of Kuramochi boundary of R except for points belonging to a set of venishing outer Kuramochi capacity.

In spite of their axiomatic attack, the most general condition for f that Beurling's theorem holds is not gained. That is, the condition that f is a Dirichlet mapping is sufficient but not necessary for Beurling's theorem. The purpose of this section is to get a new and more general condition for f(z) and R'^* to remain Beurling's theorem valid.

1. Metrizable compactifications satisfying a separability axiom

Let R^* be a metrizable compactification of R which satisfies the following condition. For any two disjoint closed sets A and B of R^*, there exists an integer n such that the 1/n -neighborhoods A_n and B_n of A and B has positive extremal distance $\lambda(A_n, B_n)$, where $\lambda(A_n, B_n)$ is defined as the extremal length of the family Γ of all curves joining $A_n \cap R$ and $B_n \cap R$. When two closed sets A and B satisfy the above condition, we say that they are <u>separated with respect to extremal distance</u>. And, when a metrizable compactification R^* satisfies the condition that any two disjoint closed sets of R^* are separated with respect to extremal distance, we say that the compactification R^* satisfies <u>the separability axiom with respect to extremal distance</u>, and denote it by R_E^*.

By definition, R_E^* is an HD-separative compactification. In connection with a quotient space of Royden compactification, we have the following theorem.

Theorem 3.5. Let R^* be a metrizable compactification of a Riemann surface R. If R^* is a quotient space of Royden compactification, then R^* satisfies the separability axiom with respect to extremal distance. Accordingly, Kuramochi compactification satisfies the separability axiom.

Before the proof, we prove a lemma.

Lemma 3.1. ([18]) Let R_D^* be Royden compactification of a Riemann surface R, and A and B be two disjoint closed sets of R_D^*, then there exists a continuous Dirichlet function u(z) such that u(z) = 1 on A and = 0 on B.

Proof For any point p of A, there exists a continuous Dirichlet function v_p such that $v_p = 1$ at p and $v_p = 0$ on B, and $0 \leq v_p \leq 1$ on R_D^*. The set $U(p) = \{z;\ v_p(z) > 1/2\}$ is an open neighborhood of p, and A is covered by a finite number of these neighborhoods $U_1, U_2, \cdots, U_n$ because A is compact. Let $v_1, v_2, \cdots, v_n$ be the corresponding functions, then, $v = v_1 + v_2 + \cdots + v_n$ is a Dirichlet function such that $v > 1/2$ on A and v = 0 on B, and min(2v, 1) is the required Dirichlet function.

Proof of the theorem. According to Constantinescu and Cornea, there exists a continuous mapping φ of R_D^* onto R^* which is identity in R. Let A_n and B_n be neighborhoods of A and B, respectively, such that their closures $\bar{A}_n$ and $\bar{B}_n$ with respect to R^* are disjoint, then the images $\varphi^{-1}(\bar{A}_n)$ and $\varphi^{-1}(\bar{B}_n)$ are disjoint closed sets in R_D^*.

Hence, there exists a continuous Dirichlet function v in R such that $v = 1$ on $A_n \cap R$ and $v = 0$ on $B_n \cap R$, and the extremal distance $\lambda(A_n, B_n) \geqq 1/D(v) > 0$. Therefore, R^* satisfies the separability axiom with respect to extremal distance.

Next, we denote by Δ_E the ideal boundary of R_E^* and consider a family Γ of all locally rectifiable curves which start from points of R and tend to Δ_E of R_E^*. Then, we have the following theorem.

Theorem 3.6. Every curve of Γ converges to a point of Δ_E except for curves belonging to a family of the infinite extremal length.

Proof Let Γ' be the family of curves c' of Γ which does not converge to a point of Δ_E, that is, for end parts $e_n = c' \cap (R - R_n)$, $\bigcap_n \bar{e}_n$ contains more than two points of Δ_E, where $\{R_n\}$ is an exhaustion of R, and the closure $\bar{e}_n$ is taken with respect to R_E^*. Suppose two points a and b belong to $\bigcap_n \bar{e}_n$, then there exist the neighborhoods V(a) and V(b) of a and b respectively such that the extremal distance between V(a) and V(b) is positive. Let $\rho|dz|$ be an admissible metric for the problem of the extremal distance $\lambda(V(a), V(b))$ such that

$$\inf_{\gamma \in \Sigma} \int_\gamma \rho|dz| > \frac{1}{2} \lambda(V(a), V(b))^{\frac{1}{2}},$$

where Σ denotes the family of all curves joining $V(a) \cap R$ and $V(b) \cap R$. Then, since $c' \in \Gamma'$ cuts both V(a) and V(b) infinitely many times,

$$\int_{c'} \rho|dz| = \infty .$$

Hence, we have, for the extremal length $\lambda_{\{c'\}}$ of the family c' of all curves which cut both V(a) and V(b) infinitely many times,

$$\lambda_{\{c'\}} \geqq \left(\inf_{\{c'\}} \int_{c'} \rho \, |dz| \right)^2 = \infty \text{ 6)}.$$

Now, since R_E^* has a countable base $\{V_n\}$ of neighborhoods, there exists V_n and V_m of $\{V_n\}$ such that $V_n \subset V(a)$, $V_m \subset V(b)$ and

$$0 < \lambda(V(a), V(b)) < \infty .$$

So, the family Γ' is exhausted by a countable number of families, Γ_{nm} of curves which cut both V_n and V_m infinitely many times. And the extremal length λ_{nm} of each Γ_{nm} is infinite. Therefore,

$$1/\lambda_{\Gamma'} \leqq \sum 1/\lambda_{nm} = 0 ,$$

that is, $\lambda_{\Gamma'} = \infty$.

This theorem is an extension of Theorem 1.1.

According to this theorem, if R is hyperbolic every Green line converges to a point of the ideal boundary of R_E^* except for Green lines belonging to a family of infinite extremal length, that is, except for Green lines belonging to a family whose Green measure is zero.

As a remark, we notice that we can define actually the extremal distance between two closed sets A and B of R_E^* as the extremal length of the family of curves both end parts of which converge to points of A and B, respectively.

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6) The auther owes this part of the proof to M. Ohtsuka.

2. GD mappings

In this paragraph we study limits and cluster sets of analytic mappings of a Riemann surface R into a Riemann surface R', whose values fall in the compactification R'^*_E. Closed sets E and E' in R' are said <u>separated in R' with respect to extremal distance</u> when there exist neighborhoods E_n and E'_n in R' of E and E', respectively, whose extremal distance $\lambda(E_n, E'_n)$ is positive. We consider an analytic mapping f which satisfies the following condition : if E and E' are closed sets separated in R' with respect to extremal distance, then $f^{-1}(E)$ and $f^{-1}(E')$ are also separated in R with respect to extremal distance. We shall call such an analytic mapping a <u>GD mapping</u>.

Theorem 3.7. A Dirichlet mapping f of a Riemann surface R into a Riemann surface R' is a GD mapping in our sense.

Previous to the proof, we prove the following lemma.

Lemma 3.2. Let G and G' be open sets in R with piecewise smooth relative boundaries. If the extremal distance $\lambda(\bar{G}, \bar{G}')$ is positive, there exists a Dirichlet function u such that it is harmonic in $R - \bar{G} \cup \bar{G}'$ and

$$u = \begin{cases} 1 & \text{on } \bar{G} \\ 0 & \text{on } \bar{G}' . \end{cases}$$

Proof Let $\{R_n\}$ be a regular exhaustion of R and put $\lambda_n = \lambda(\bar{G}\cap R_n, \bar{G}'\cap R_n)$. Then the sequence $\{\lambda_n\}$ decreases to λ by Suita's lemma [33], and we have

$$\lambda_n = \frac{1}{D(u_n)} ,$$

where u_n is a harmonic function in $R_n - \bar{G} \cup \bar{G}'$ such that

$$u_n = \begin{cases} 1 & \text{on } \bar{G} \cap \bar{R}_n \\ 0 & \text{on } \bar{G}' \cap \bar{R}_n , \end{cases}$$

$$\frac{\partial u_n}{\partial \nu} = 0 \qquad \text{on the rest of } \partial R_n .$$

Hence, Dirichlet integrals $D(u_n)$ are uniformly bounded by the extremal length $\lambda^{-1} < \infty$.($\lambda < \infty$ because both G and G' are open.

Now, for $m > n$, we have, for inner products,

$$(du_n, d(u_n - u_m))_{R_n - \bar{G} \cup \bar{G}'} = 0$$

and

$$(du_n, du_m)_{R_n - \bar{G} \cup \bar{G}'} = D_{R_n - \bar{G} \cup \bar{G}'}(u_n).$$

Hence,

$$D_{R_n - \bar{G} \cup \bar{G}'}(u_n - u_m) \leqq D_{R_m - \bar{G} \cup \bar{G}'}(u_m) - D_{R_n - \bar{G} \cup \bar{G}'}(u_n).$$

Therefore, u_n converges to the desired function in Dirichlet norm and uniformly on every compact sets in $R - \bar{G} \cup \bar{G}'$.

Proof of the theorem. Let f be a Dirichlet mapping of R into R', and E and E' be disjoint closed sets separated in R' with respect to extremal distance, that is, there exist neighborhoods E_n and E_n' of E and E' in R' respectively, whose extremal distance $\lambda(\bar{E}_n, \bar{E}_n')$ is positive, where we can assume without loss of generality that the relative boundaries E_n and E_n' are piecewise smooth (cf., for instance, [28] p.289). Then, by the above lemma, there exists a harmonic Dirichlet function u in R' such that

$$u = \begin{cases} 1 & \text{on } \bar{E}_n \\ 0 & \text{on } \bar{E}_n' , \end{cases}$$

where the closures are taken in R'. This u is extendable continuously on Royden Compactification R'^*_D, and we put, for extended u,

$$E^* = \left\{ p \; ; \; p \in R'^*_D \quad u(p) \geqq 1 - \varepsilon \right\}$$

$$E'^* = \left\{ p \; : \; p \in R'^*_D \quad u(p) \leqq \varepsilon \right\},$$

then E^* and E' are disjoint closed sets in R'^*_D containing E_n and E'_n, respectively.

Since f is a Dirichlet mapping, f is extendable continuously to a mapping of R^*_D into R'^*_D. Denoting the extended mapping by f again, $f^{-1}(E^*)$ and $f^{-1}(E'^*)$ are disjoint closed sets in R^*_D. So, by Lemma 3.1, there exists a continuous Dirichlet function v on R^*_D such that

$$v = \begin{cases} 1 & \text{on } f^{-1}(E^*) \\ 0 & \text{on } f^{-1}(E'^*) \ . \end{cases}$$

Hence, the extremal distance $\lambda(f^{-1}(E^*) \cap R, \ f^{-1}(E'^*) \cap R)$ is positive. Therefore, by monotonicity of extremal distance, $f^{-1}(E)$ and $f^{-1}(E')$ are disjoint closed sets in R separated with respect to extremal distance.

We now study boundary behaviors of GD mappings. For this purpose, it is suitable to consider Kuramochi compactification R^*_N of R as a domain of definition, and a metrizable compactification R'^*_E satisfying the separability axiom with respect to extremal distance as a range of cluster values of the GD mapping. At first, we prove the following theorem concerning limits along curves of a GD mapping.

Theorem 3.8. Let f be a GD mapping and Γ be the family of all curves which start from points of R and tend to the ideal boundary. Then, f has limits in R'^*_E along every curve of Γ except for curves belonging to a family of the infinite extremal length.

Proof Assume f has not a limit along a curve c of Γ. The cluster set of f in R_E^* along c contains more than two points a and a'. And there exist neighborhoods V(a) and V(a') whose extremal distance is positive. We put $G = f^{-1}(V(a))$ and $G' = f^{-1}(V(a'))$. Then, the extremal distance $\lambda(G, G')$ may be assumed positive by taking V(a) and V(a') suffiently small. Since the curve c cuts both G and G' infinitely many times, we conclude, as in the proof of Theorem 3.6, that the family $\Gamma_{aa'}$ of curves which cut both G and G' infinitely many times has the infinite extremal length.

By definition, R_E^* is metrizable and has a countable base of neighborhoods. If we denote by Γ_o the family of all curves of along which f has not limits, Γ_o is exhausted by a countable number of such families as $\Gamma_{aa'}$. Therefore the extremal length of Γ_o is infinite, q.e.d.

Now, we are going to extend Beurling's theorem of Fatou type. We consider Kuramochi compactification R_N^* of R and Kuramochi boundary Δ_N, and denote by Δ_1 the set of minimal points of Δ_N. $\Delta_o = \Delta_N - \Delta_1$ is of Kuramochi capacity zero. To each point p of Δ_1 we associate the filter base $\mathcal{G}_p$ of open sets stated before. For each $\mathcal{G}_p$ we defien the cluster set $f^{\vee}(p)$ of a QD mapping f of R into R' whose values fall in the compactification ${R'}_E^*$ as

$$f^{\vee}(p) = \bigcap_{G \in \mathcal{G}_p} \overline{f(G)} \quad ,$$

where the closure is taken in ${R'}_E^*$. $f^{\vee}(p)$ is a non-void closed set in ${R'}_E^*$. We call $f^{\vee}(p)$ fine cluster set of f at p, and when $f^{\vee}(p)$ is one point we call it a fine limit.

Our extension of Beurling's theorem is as follows.

Theorem 3.9. A GD mapping of R into R' has a fine limit in R_E^* quasi-everywhere on Δ_N.

Proof Let p be a point of Δ_1 at which f has not a fine limit, that is, the fine cluster set $f^V(p)$ contains more than two points a and a'. By definition of R'^*_E, there exist disjoint neighborhoods V(a) and V(a') of a and a' respectively, whose extremal distance is positive. We put $G = f^{-1}(V(a))$ and $G' = f^{-1}(V(a'))$. For any open set U of $\mathcal{G}_p$, $\overline{f(U)} \cap V(a) \neq \emptyset$ and $\overline{f(U)} \cap V(a') \neq \emptyset$. And since V(a) and V(a') are open sets, $f(U) \cap V(a) \neq \emptyset$ and $f(U) \cap V(a') \neq \emptyset$. Hence, $G \cap U \neq \emptyset$ and $G' \cap U \neq \emptyset$. And, by definition of GD mappings, the extremal distance $\lambda(\overline{G}, \overline{G}')$ may be assumed positive by taking V(a) and V(a') sufficiently small. Then, there exists a Dirichlet function u such that

$$u = \begin{cases} 1 & \text{on } \overline{G} \\ 0 & \text{on } \overline{G}' . \end{cases}$$

This u can not have a fine limit at p. On the other hand, Dirichlet functions in R have fine limits at every point of Δ_N except for points belonging to a set of vanishing outer Kuramochi capacity. Hence, the set

$$E = \{ q \in \Delta_1 ; \ f^V(q) \cap V(a) \neq \emptyset \ \text{ and } \ f^V(q) \cap V(a') \neq \emptyset$$

must be of zero outer Kuramochi capacity.

Since R'^*_E is metrizable, R'^*_E has a countable base of neighborhoods. Therefore, the set of points at which f does not have fine limits is covered by a countable number of sets such as E above, and is of zero outer Kuramochi capacity. This completes the proof.

§ 4 Evans potential

We refer first to the Evans potential for a compact set with the infinite transfinite diameter on Kuramochi compactification of an arbitrary Riemann surface and next for an ideal boundary of a parabolic Riemann surface, and finally another type of potential like Evans' one for a set of capacity zero on an arbitrary metrizable compactification of any Riemann surface.

1. Evans potential for a compact set with the infinite transfinite diameter on Kuramochi compactification. For a compact set K on $R_o^* = R_N^* - K_o$ we define the transfinite diameter D(K) as follows.

$$D_n(K) = \frac{1}{{}_nC_2} \inf_{p_1 \cdots p_n \in K} \sum_{i<j}^{1 \cdots n} N(p_i, p_j) .$$

Then, by the same argument as in the case of Green function ([32]), we know the limit

$$\lim_{n \to \infty} D_n(K) = D(K) \qquad (\leqq \infty)$$

exists, which is called the transfinite diameter of K.

In the case that the transfinite diameter of K is infinite, we can construct the following Evans potential.

Theorem 4.1. Let K be a compact set with the infinite transfinite diameter on R_o^*. Then there exists a potential

$$p(z) = \int_K N(z,p)\, d\mu(p)$$

such that

1) $p(z)$ is harmonic in $R_o - K$

2) $p(z) = 0$ on K_o

3) $\lim\limits_{R_0 \ni z \to K} p(z) = \infty$.

4) $D(\min(p(z), M)) < \infty$

We remark that a compact set of vanishing Kuramochi capacity is not necessarily of the infinite transfinite diameter.

2. In the case of a parabolic Riemann surface, the existence of the Evans potential was proved by Kuramochi[12] and M. Nakai[24].

Theorem 4.2. (Nakai) Let R be a parabolic Riemann surface. Then, there exists a positive harmonic function p(z) on R_0 such that

1) p(z) is continuously 0 on ∂K_0

2) $\lim p(z) = \infty$ when z tends to the ideal boundary of R

3) $\int_{\partial K_0} dp^* = 2\pi$ when K_0 is positively oriented, and

$D(\min(p(z), M)) \fallingdotseq 2\pi M$.

3. For an arbitrary Riemann surface R we construct another type of potential. It is constructed, for a closed set E of vanishing capacity of metrizable compactification R^* of R. Let E be a closed set of R^* such that $E \cap K_0 = \emptyset$, E_n be the closed 1/n -neighborhood of E and $E_n' = R \cap E_n$. We may assume $\partial E_n'$ is piecewise analytic, modifying it if necessary. Let $1_{E_n'} = \omega_n(z) = \omega(z, E_n')$ be Kuramochi's capacitary potential of E_n', that is, ω_n be a positive harmonic function in $R - K_0 - E_n'$ such that

$$\omega_n = \begin{cases} 0 & \text{on } \partial K_0 \\ 1 & \text{on } \partial E_n' \end{cases}$$

and ω_n has the minimal Dirichlet integral. We extend ω_n to be 0 on K_0, and 1 on E_n'. If the Dirichlet integrals $D(\omega_n)$ are uniformly bounded, the sequence $\{\omega_n\}$ converges in Dirichlet norm and hence converges uniformly on every compact set of R_o. The limit function ω_E is harmonic in $R_o - E$ and has a finite Dirichlet integral $D(\omega_E)$. We call $D(\omega_E)$ the capacity of E. This is a weak capacity in the sense of Brelot [3].

We assume, here, the capacity of E zero. Then, the function ω_E is zero and there exists a sequence of $\omega_n(z)$ which converges to zero. We can choose a subsequence of $\{\omega_n(z)\}$, which we write $\{\omega_n(z)\}$ again, such that $\sum_{n=1}^{\infty} \omega_n(z)$ converges. The limit function $p(z) = \sum_{n=1}^{\infty} \omega_n(z)$ is continuous in $R - E$ and has a finite Dirichlet integral.

In fact, by the definition of capacity, there exists a sequence $\{E_n\}$ of neighborhoods of E such that the capacity of E_n is smaller than $1/_{2}n$. Putting $p_m(z) = \sum_{n=1}^{m} \omega_n(z)$, the potentials, $p_m(z)$ have uniformly bounded energies (Dirichlet integrals), converges in Dirichlet norm and converges uniformly on every compact sets K on R such that $K \cap E_n = \emptyset$ for some n. Hence, the limit function p(z) is continuous in $R - E$ and has finite energy. The set $G_M = \{z \in R \ ; \ p(z) > M$ is consequently open.

p(z) tends to infinity when z tends to E because, for any number M, $p(z) > p_n(z) \geqq M$ in E_n' when $n > M$. By the above fact, we know the open set $R^* - \overline{(R - G_M)}$ contains E, where closure is taken in R^*.

Next we show the level curve $C_M = \{ z \in R \ ; \ p(z) = M \}$ is piece-wise analytic, that is, it consists of a countable number of analytic curves and arcs. Put $m = [M]$. Then $p_{m+1} > M$ in E_{m+1} because $\omega_1 = \omega_2 = \cdots\cdots = \omega_M = \omega_{m+1} = 1$ in E_{m+1}, and $C_M \cap E_{m+1} = \emptyset$. We write

$$C_M = C_M \cap (E_{m-1} - E_m) + C_M \cap (E_{m-2} - E_{m-1}) + \cdots\cdots + C_M \cap (E_1 - E_2) + C_M \cap (R - E_1)$$

and

$$p(z) = p_j(z) + \sum_{n=j+1}^{\infty} \omega_n(z) = p_j(z) + q_j(z).$$

Then, in $E_j - E_{j+1}$, $p_j(z)$ is a constant $= j$ and $q_j(z)$ is harmonic. Hence, the curve $C_M \cap (E_j - E_{j+1})$ is determined by the equality $q_j(z) = M - j$, and so it is piecewise analytic. Since C_M consists of countable parts $\{C_M \cap (E_j - E_{j+1})\}$, it is piecewise analytic.

Theorem 4.3. Let E be a closed set of R^* with vanishing capacity. There exists a continuous potential $p(z)$ in R with finite energy (Dirichlet integral), such that

$$\lim_{R \ni z \to E} p(z) = \infty$$

and the level curves are piecewise analytic.

§ 5 Extension of Beurling's theorem of Riesz type

1. The purpose of this section is to extend the following Beurling's theorem of Riesz type [1] .

Let $f(z)$ be a meromorphic function in the unit disk $|z|<1$ and F be the covering surface generated by $w = f(z)$ over w-sphere W. Let a be a point of W and F_r be the part of F which lies over the disk centered at a with radius r. Let $A(r)$ be the spherical area of F_r.

If

$$\delta(a) = \varlimsup_{r\to 0} \frac{A(r)}{\pi r^2} < \infty ,$$

a is called an ordinary value of f(z) in Beurling's sense. If the spherical area of F is finite, then almost every point of W is an ordinary value of f(z).

Theorem 5.1. (Beurling) Let f(z) be a meromorphic function in $|z|<1$ for which the spherical area of F is finite. Let a be an ordinary value of f(z) in Beurling's sense, then the set E of $e^{i\theta}$ such that $\lim_{r\to 1} f(re^{i\theta}) = a$ is of outer capacity zero.

This theorem was generalized in various directions by M. Tsuji, A. Pfluger and Z. Kuramochi. Pfluger[30] replaced the radial limit $\lim_{r\to 1} f(re^{i\theta})$ in the theorem by the asymptotic value along an arc ending at $e^{i\theta}$. Kuramochi [17] extended Beurling's theorem to the case that f(z) is an analytic mapping f of a Riemann surface R into another R'. He replaced the radial limit by the cluster set of f with respcet to the contact set, and generalized the ordinary value in Beurling's sense to a set of points in a parametric disk of R' satisfying an analogous condition. Our purpose is to treat a set containing ideal boundary points as a generalization of the ordinary value in Beurling's sense. For this purpose we are obliged to set an assumption that E' is of capacity zero.

2. Let R_N^* be Kuramochi compactification of R and R'^* be a compactification of R'. At first we put no restriction upon R'^*. We consider a continuous mapping f of R into R', and define for $p\in \Delta_1$ the fine cluster set $f^V(p)$ in R'^* as follows.

$$f^{\vee}(p) = \bigcap_{G \in \mathcal{G}_p} \overline{f(G)} ,$$

where closure is taken in R'^*.

Constantinescu-Cornea[4] proved the following

Lemma 5.1. For an open set G' in R'^* which contains $f^{\vee}(p)$, $f^{-1}(G')$ belongs to $\mathcal{G}_p$.

Now, for an open set G in R, we consider the set $\Delta_1(G)$ of all points of Δ_1 at which $R - G$ is thin. For Kuramochi capacity of $G \cup \Delta_1(G)$, we prove the following lemma.

Lemma 5.2. For an open set G in R, if inner Kuramochi capacity of $G \cup \Delta_1(G)$ is finite it is equal to Kuramochi capacity of G.

Proof Let $C_i(G \cup \Delta_1(G))$ be inner Kuramochi capacity of $G \cup \Delta_1(G)$, then there exists an increasing sequence $\{K_n\}$ of compact sets in $G \cup \Delta_1(G)$ such that every point of G is an inner point of at least one of $\{K_n\}$ and Kuramochi capacity $C(K_n)$ of K_n tends to $C_i(G \cup \Delta_1(G))$.

Let p^{μ_n} be an equilibrium potential of K_n, then the sequence $\{p^{\mu_n}\}$ is monotone increasing and the energies (Dirichlet integrals). $\{\|dp^{\mu_n}\|^2\}$ are uniformly bounded. So, according to [4], $\{\mu_n\}$ converges to a canonical mass distribution μ such that

$$\lim_{n\to\infty} \| dp^{\mu_n} - dp^{\mu} \| = 0$$

and

$$\lim_{n\to\infty} p^{\mu_n} = p^{\mu} .$$

Hence, the energy

$$\|\mu\|^2 = \|dp^{\mu}\|^2 = \lim_{n\to\infty} \| dp^{\mu_n}\|^2 = \lim_{n\to\infty} C(K_n) = C_i(G \cup \Delta_1(G)).$$

On the other hand, $p^{\mu} = \lim_{n\to\infty} p^{\mu_n} = 1$ in G, and since p^{μ} is a Dirichlet function in R and fine continuous quasi-everywhere on Δ, $p^{\mu} = 1$ quasi-everywhere on $G \vee \Delta_1(G)$. Let s be a positive full-superharmonic function in R, which is not smaller than 1 in G. Then, s is fine continuous quasi-everywhere on Δ and hence $s \geqq 1$ quasi-everywhere on $\Delta_1(G)$. So, by domination principle,

$$s \geqq p^{\mu_n},$$

and

$$s \geqq p^{\mu} = \lim_{n\to\infty} p^{\mu_n}.$$

This fact shows that p^{μ} is the minimum among non negative full-superharmonic functions which are not less than 1 on G, that is, p^{μ} is the equilibrium potential of G ([4] p. 187). We conclude

$$C_1(G \cup \Delta_1(G)) = C(G).$$

Corollary. Let $\{G_n\}$ be a sequence of open sets in R and E be a set contained in $\Delta_1(G_n)$ for all G_n. If Kuramochi capacity $C(G_n)$ tends to zero, inner Kuramochi capacity $C_1(E)$ of E is zero.

Proof Let K be a compact set in E. Then, $0 \leqq p^{\mu_K} \leqq p^{\mu_{G_n}}$ for all n by domination principle, where μ_K and μ_{G_n} denote the equilibrium mass distribution for K and G_n, respectively. If $p^{\mu_{G_n}}$ tends to zero p^{μ_K} must be zero, that is, $\| dp_1^{\mu_K} \| = 0$ and $C(K) = 0$. Since K is arbitrary we conclude $C_1(E) = 0$.

Lemma 5.3. Let G be an open set in R_0, whose relative boundary consists of piecewise smooth curves. Then, $2\pi C(G)$ is dominated by the reciprocal of the extremal distance between K_0 and $\bar{G}$.

Proof Let $\{R_n\}$ be a regular exhaustion of R, and ω_n be the solution of Dirichlet principle in R_n such that

$$\omega_n = \begin{cases} 1 & \text{on } \overline{G} \cap \overline{R}_n \\ 0 & \text{on } K_o \end{cases}$$

$$\frac{\partial \omega_n}{\partial \nu} = 0 \qquad \text{on } \partial R_n - \overline{G}.$$

Then ω_n tends to ω_G in Dirichlet norm and uniformly on every compact set in $R_o - \overline{G}$. The extremal distance $\lambda(K_o, \overline{G})$ between K_o and $\overline{G}$ equals the reciprocal of Dirichlet integral $D(\omega_G)$. But, for any compact set K in G, $\frac{1}{2\pi} \lambda(K_o, \overline{G})^{-1}$ is greater than C(K). Hence, we have

$$\frac{1}{2\pi} \lambda(K_o, \overline{G})^{-1} \geqq C(G).$$

3. First, we treat the case that R' is hyperbolic and R'^* is a metrizable compactification without any more restriction. In the following three case, we consider always Kuramochi compactification R_N^* of R.

Let E' be a closed set on R'^*, which is of capacity 0 in the sense of § 4.3. Then, there exists a continuous potential in R' with finite energy such that

$$\begin{cases} p(w) = 0 & \text{on } K'_o \\ \lim\limits_{R' \ni w \to E'} p(w) = \infty , \end{cases}$$

and the level curve $C'_M = \{ w \in R' \ ; \ p(w) = M \}$ is piecewise analytic.

We consider an analytic mapping $w = f(z)$ of R into R'. We take K_o and K'_o as $f(K_o) \subset K'_o$ and put $G'_M = \{ w \in R' \ ; \ M < p(w) \}$ and

$G'_{MM_o} = \{w \in R' \ ; \ M_o < p(w) < M\}$ for a fixed positive number M_o. For the sets $G_M = f^{-1}(G'_M)$ and $G_{MM_o} = f^{-1}(G'_{MM_o})$, we denote by ~~$D_{G_M}(u)$ and~~ $D_{G_{MM_o}}(u)$ the Dirichlet integrals of $u = p\circ f$ over ~~G_M and~~ G_{MM_o}. ~~respectively.~~

Theorem 5.2. Let E' be a closed set of vanishing capacity on R'^*. If an analytic mapping $w = f(z)$ satisfies the condition that

$$\lim_{M\to\infty} \frac{D_{G_{MM_o}}(u)}{M^2} = 0 , \qquad u = p\circ f$$

Then, the set $E = \{p \in \Delta_1 \ ; \ f^{\vee}(p) \subset E'\}$ has vanishing inner Kuramochi capacity.

Proof Let E'_n be the $1/n$ -neighborhood of E'. For given M, if n is sufficiently large $p(z) > M$ in $E'_n \cap R'$, and $f^{-1}(E'_n) \subset G_M$. Since E'_n is an open set containing E', we have

$$f^{-1}(E'_n) \in \bigcap_{p\in E} \mathcal{G}_p ,$$

and

$$G_M \in \bigcap_{p\in E} \mathcal{G}_p .$$

This shows that

$$\Delta_1(G_M) \supset E.$$

We consider a family Γ_{MM_o} of curves c which join the sets $\{z \in R \ ; \ u(z) \geq M\}$ and $\{z \in R \ ; \ u(z) \leq M_o\}$, and denote by λ_{MM_o} its extremal length. By the assumption of the theorem $D_{G_{MM_o}}(u)$ is finite, and

$$\rho|dz| = \begin{cases} \dfrac{|\mathrm{grad}\ u|}{\sqrt{D_{G_{MM_0}}(u)}}\,|dz| & \text{in } G_{MM_0} \\ 0 & \text{elsewhere} \end{cases}$$

is admissible for the problem of λ_{MM_0}. So we have

$$\lambda_{MM_0} \geqq \Big(\inf_{c\in\Gamma_{MM_0}} \int_c \rho|dz|\Big)^2 = \frac{1}{D_{G_{MM_0}}(u)}\Big(\int_c |\mathrm{grad}\ u|\ |dz|\Big)^2 \geqq \frac{(M - M_0)^2}{D_{G_{MM_0}}(u)} .$$

From the assumption of the theorem, we have

$$\lim_{M\to\infty} \lambda_{MM_0} = \infty .$$

By Lemma 5.3,

$$c(G_M) \leqq \frac{1}{2\pi} \lambda(K_0, \bar{G}_M)^{-1} \leqq 1/2\pi\lambda_{MM_0} .$$

So, by the Corollary to Lemma 5.2, we conclude that inner Kuramochi capacity of E vanished.

4. Secondly, we treat the case that R'^* is Kuramochi compactification R'^*_N ($R'^*_0 = R'^*_N - K'_0$) of a hyperbolic Riemann surface R'. Let E' be a closed set on R'^*_N of the infinite transfinite diameter. Then, as we have shown, there exists Evans potential $p(w)$ such that

1) $p(w)$ is harmonic in $R'_0 - E'$

2) $p(w) = 0$ on K_0 and $\lim_{R'_0 - E' \ni w \to E'} p(w) = \infty$.

3) $D(\min(p(w), M)) < \infty$

As Constantinescu and Cornea pointed out, there are some difficulties in potential theory on Kuramochi compactification, which is due to existence off non-minimal points. We notice here that a set of vanishing Kuramochi capacity is not necessarily of the infinite transfinite diameter. For instance, it is possible that $\sup_{z\notin R} N(z, q)$

is finite for a non-minimal point q, and $N(q,q) < \infty$ (cf.[13]). For such a point, Kuramochi capacity is zero but the transfinite diameter can not be infinite.

For a closed set E' of the infinite transfinite diameter, we have the similar theorem as that of paragraph 3.

Theorem 5.3. Let E' be a closed set of the infinite transfinite diameter on R'^*_N. If an analytic mapping $w = f(z)$ satisfies the condition that

$$\lim_{M\to\infty} \frac{D_{G_{MM_0}}(u)}{M^2} = 0 , \qquad u = p\circ f$$

then, the set $E = \{ p \in \Delta_1 \ ; \ f^{\vee}(p) \subset E' \}$ has vanishing inner Kuramochi capacity.

But in this case, $u = p\circ f$ is harmonic in R_0 outside a set of zero points of $f'(z)$ and points of $f^{-1}(E')$.

This gives an extension of Pfluger's theorem.

Let R' be the Riemann sphere and E' be the origin $w = 0$. We take $|w| \geq 1$ as K'_0. In this case Evans potential $p(w)$ is equal to N-Green function $N(w, 0) = \log \frac{1}{r}$, where $w = re^{i\theta}$. We have

$$D_{G_{MM_0}}(u(z)) = \iint_{G_{MM_0}} |\operatorname{grad} N(w,0)|^2 \, |f'(z)|^2 dxdy$$

$$= \iint_{\substack{e^{-M} < r < e^{-M_0} \\ 0 \leq \theta < 2\pi}} \frac{1}{r^2} n_f(w) \, r \, dr d\theta ,$$

where $n_f(w)$ denotes the number of w-points of f(z). And

$$\frac{D_{G_{MM_0}}(u)}{(M - M_0)^2} = \frac{1}{(M - M_0)^2} \iint \frac{1}{r^2} n_f(w) \, r \, dr d\theta$$

$$= \frac{1}{(\log \frac{r_o}{r})^2} \iint \frac{1}{r^2} n_f(w)\, r\, dr d\theta .$$

Putting

$$S(t) = \iint_{\substack{r<t \\ 0\leq\theta<2\pi}} n_f\, r\, dr d\theta ,$$

we have

$$\frac{D_{G_{MM_o}}(u)}{(M - M_o)^2} = \frac{1}{(\log \frac{r_o}{r})^2} \int_{\gamma}^{\gamma_o} \frac{dS(t)}{t^2} .$$

It is sufficient for λ_{MM_o} to tend to infinity that

$$\lim_{r\to 0} \frac{1}{(\log \frac{r_o}{r})^2} \int_{\gamma}^{\gamma_o} \frac{dS(t)}{t^2} = 0.$$

But, in this case, it is possible to give a sharper condition.

Take r_1 and r_2 as $r< r_1< r_2< r_o$ and write

$\lambda_{r_1 r_2} = \lambda_{e^{-\gamma_1} e^{-\gamma_2}}$, then $\lambda_{MM_o} \geq \lambda_{r_1 r_2}$. Following Pfluger,

we calculate the integral by integration by parts.

$$\frac{1}{(\log \frac{r_2}{r_1})^2} \int_{r_1}^{r_2} \frac{dS(t)}{t^2} \leqq \frac{S(r_2)}{r_2^{\,2}(\log \frac{r_2}{r_1})^2} + \frac{2}{(\log \frac{r_2}{r_1})^2} \int_{r_1}^{r_2} \frac{S(t)}{t^2}\frac{1}{t}\, dt$$

We assume here

$$S(t) = o(t^2 \log \frac{1}{t}) .$$

The first term on the right side of the inequality tends to 0 with r_1.

So, the first term $< \varepsilon$ if r_1 is sufficiently small. For the second term, since

$$\frac{S(t)}{t^2 \log \frac{1}{t}} = \varepsilon' \qquad (\varepsilon' \to 0 \text{ for } t \to 0)$$

when $r_1 < t < r_2$ for sufficiently small r_1 and r_2,

$$\int_{r_1}^{r_2} \frac{S(t)}{t^2} \frac{1}{t} dt = \int_{r_1}^{r_2} \varepsilon' \log \frac{1}{t} d\log t = \frac{\varepsilon'}{2} \left\{ (\log r_1)^2 - (\log r_2)^2 \right\}$$

We have

$$\frac{2}{(\log \frac{r_2}{r_1})^2} \int_{r_1}^{r_2} \frac{S(t)}{t^2} \frac{1}{t} dt \to \varepsilon' \quad \text{for } r_1 \to 0.$$

Hence $\lambda_{r_1 r_2}$ is arbitrarily large for sufficiently small r_1 and r_2.

So, we conclude that λ_{MM_0} tends to infinity when M tends to ∞ .

Our theorem is idential with Pfluger's one in the case of the unit disk. To show this we prove the following

Lemma 5.4. (Pfluger [30]) Let $w = f(z)$ be a meromorphic function in the unit disk $|z| < 1$ such that

$$D_f = \iint_{|z|<1} \left(\frac{|f'|}{1 + |f|^2} \right)^2 dxdy < \infty .$$

If $f(z)$ tends to 0 along a curve c terminating at $e^{i\theta}$, then,

$$\lim_{z \to e^{i\theta}} |f(z)| = 0$$

along every curve terminating at $e^{i\theta}$.

Proof Let Γ be the family of concentric circular arcs, γ in $|z| < 1$ centered at $e^{i\theta}$. Then, extremal length λ_Γ of Γ is zero.

Since $D_f < \infty$,

$$\rho = \frac{1}{\sqrt{D_f}} \frac{|f'|}{1 + |f|^2}$$

is admissible for the problem of the extremal length λ_Γ. Hence, by the definition of extremal length,

$$\inf_{\gamma \in \Gamma} \int_\gamma \frac{|f'|}{1 + |f|^2} \quad |dz| = 0.$$

Hence there exists a sequence $\{\gamma_n\}$ in Γ for which

$$\lim_{n\to\infty} \int_{\gamma_n} \frac{|f'|}{1 + |f|^2} |dz| = 0.$$

This means that the spherical length of the image of γ_n tends to 0. Every curve c' terminating at $e^{i\theta}$ intersects γ_n at a point z_n. And $f(z_n)$ tends to 0. This means that

$$\lim_{\substack{z\to e^{i\theta}\\ z\in c'}} |f(z)| = 0.$$

This lemma shows that the fine cluster set can be replaced by the asymptotic value in the case of the unit disk.

Let E be a set of points $\{e^{i\theta}\}$ such that there exists a curve terminating at $e^{i\theta}$ along which $\lim_{z\to e^{i\theta}} f(z) = 0$. By above Lemma,

$$\lim_{z\to e^{i\theta}} |f(z)| = 0$$

along every curve c' terminating at $e^{i\theta}$. This means that the curve c' cuts G_M infinitely many times. But the extremal distance between G_M and the small disk K centered at z = 0 tends to infinitely with M. That is, the extremal length of the family of all curves which join

K and points o E is infinite. This shows that the outer capacity of E is zero (cf. [30]). Thus, we have Pfluger's theorem.

Theorem 5.4. (Pfluger [30]) Let $f(z)$ be a meromorphic function in $|z|<1$ such that the spherical area of F is finite. If $f(z)$ satisfies the condition that

$$S(t) = o(t^2 \log \frac{1}{t}) \qquad \text{for } t \to 0,$$

then the set of points on $|z| = 1$ at which $f(z)$ has asymptotic value 0 is of outer capacity zero.

We give a modification of Theorem 5.3.

Let $p(w)$ be the Evans potential for the set E' and $p^*(w)$ be its conjugate harmonic function. We use $p + ip^*$ as a local parameter.

Theorem 5.5. Let $w = f(z)$ be an analytic mapping of R into R'. If

$$\int_{p(w)=M} n_f(w)\, dp^* = o(M) \qquad \text{for } M \to \infty ,$$

the set E has vanishing inner Kuramochi capacity.

Proof As we have already shown,

$$\lambda_{M_1 M_o} \geqq \left(\frac{D_{G_{MM_o}}(u(z))}{(M_1 - M_o)^2} \right)^{-1} = \left(\frac{1}{(M_1 - M_o)^2} \iint n_f(w)\, dp dp^* \right)^{-1}$$

$$= \left(\frac{1}{(M_1 - M_o)^2} \int_{M_o}^{M_1} dp \int_{p(w)=M} n_f(w)\, dp^* \right)^{-1}$$

for $M_o < M < M_1$. If

$$\int_{p(w)=M} n_f(w)\, dp^* = o(M) \qquad \text{for } M \to \infty ,$$

there exists a number L for any $\varepsilon > 0$, such that

$$\int_{p(w)=M} n_f(w)\, dp^* < \varepsilon M \qquad \text{for } M > L.$$

Hence,

$$\iint n_f(w)\, dp dp^* = \int_{M_o}^{M_1} dp \int_{p=M} n_f(w)\, dp^* < \varepsilon M_1 \int_{M_o}^{M_1} dp = \varepsilon M_1(M_1 - M_o)$$

for $M_1 > M > M_o > L$. So, we have

$$\lambda_{M_1 M_o} > \left\{ \frac{\varepsilon M_1(M_1 - M_o)}{(M_1 - M_o)^2} \right\}^{-1} > \frac{2}{\varepsilon}$$

for $M_1 \geq 2M_o$. ε being arbitrary, $\lambda_{M_1 M_o}$ tends to infinity with M_1 and $\lim_{M_1 \to \infty} \lambda(K_o, \tilde{G}_{M_1}) \geq \lim_{M_1 \to \infty} \lambda_{M_1 M_o} \equiv \infty$: And we conclude that E has vanishing inner Kuramochi capacity.

5. Finally, we treat the case that R' is parabolic. Let E' be a closed set of vanishing Kuramochi capacity on R'^*_N. Then, R' - E' is a parabolic Riemann surface and we denote it by R' again. There exists the Evans potential p(w) such that

1) p(w) is harmonic in $R' - K'_o$ and continuous in R'
2) $p(w) = 0$ on K'_o
3) $\lim p(w) = \infty$ for $z \to$ ideal boundary
4) $\int_{\partial K'_o} dp^* = 2\pi$ when $\partial K'_o$ is positively oriented, and

$D(\min(p(w), M)) = 2\pi M.$

We use this potential to obtain sharper evaluation of the formula $D_{G_{MM_o}}(u) \big/ M^2$ in Theorem 5.3.

Let $p^*(w)$ be the conjugate harmonic function of $p(w)$ and put $g(w) = p(w) + ip^*(w)$. Then $\zeta = \psi(w) = e^{-g(w)}$ maps R' with a countable number of slits, piecewise one-to-one and conformally, onto the unit disk $|\zeta|<1$ with a countable number of slits ([25], [31]).

Denote by $\mathcal{D}_{MM_0}$ the Dirichlet integral of $\psi\circ f(z)$ over the set $\{z\in R\ ;\ M<|\psi\circ f(z)|< M_0\}$, and put $\mathcal{D}_{M_0} = \lim_{M\to 0} \mathcal{D}_{MM_0}$. Then we have

Theorem 5.7. We map R into $|\zeta|<1$ by the mapping $\zeta = \psi\circ f = \exp(-p\circ f - ip^*\circ f)$. If

$$\varliminf_{M_0\to 0} \mathcal{D}_{M_0}\Big/ M_0^2 < \infty ,$$

then the set $E = \{p\in\Delta_1\ ;\ f^{\vee}(p)\subset \Delta'_N\}$ has vanishing inner Kuramochi capacity.

Proof By definition,

$$\mathcal{D}_{MM_0} = \iint\left|\frac{d\psi\circ f}{dz}\right|^2 dxdy = \iint|\psi'(w)|^2\ |f'(z)|^2\ dxdy$$

$$= \iint|\psi'(w)|^2\ n_f(w)\ dudv \qquad (\ w = u + iv\ ).$$

For evaluation of $\mathcal{D}_{MM_0}$, we have

$$(1)\quad M^2\iint|g'(w)|^2\ n_f\ dudv < \iint|g'(w)|^2|e^{-g(w)}|^2\ n_f\ dudv$$

$$= \mathcal{D}_{MM_0} < M_0{}^2\iint|g'(w)|^2\ n_f\ dudv,$$

where integrals are taken over the set $\{w\in R';\ -\log M_0 < p(w) < -\log M\}$. And

$$(2)\quad \iint|g'(w)|^2\ n_f\ dudv = \iint|\text{grad } p(w)|^2\ n_f\ dudv$$

$$= \iint|\text{grad } p\circ f|^2|f'(z)|^2\ dxdy$$

$$= D_{MM_0}(\ p\circ f\ ).$$

Here, by assumption

$$\mathcal{D}_{M_o} = \lim_{M\to 0} \mathcal{D}_{MM_o} < \infty \quad \text{and} \lim_{M_o\to 0} \frac{\mathcal{D}_{M_o}}{M_o^2} = k < \infty .$$

By (1) and (2), we have

$$\left(\frac{M}{M_o} \right)^2 D_{MM_o}(p) < \frac{\mathcal{D}_{MM_o}}{M_o^2} \leqq \frac{\mathcal{D}_{M_o}}{M_o^2} .$$

Setting $M = \frac{1}{2} M_o$, we have

$$\frac{1}{4} D_{MM_o} \leqq \frac{\mathcal{D}_{M_o}}{M_o^2} .$$

And, since there exists a monotone decreasing sequence $\{M_n\}$ such that

$$M_n \leqq \frac{M_{n-1}}{2}, \qquad \lim_{n\to\infty} \frac{\mathcal{D}_{M_n}}{M_n^2} = k$$

and

$$\frac{\mathcal{D}_{M_n}}{M_n^2} < k + \varepsilon_n \quad (\varepsilon_n \downarrow 0),$$

we obtain, by putting $D_n = D_{\frac{1}{2}M_n, M_n}$,

$$D_n < 4(k + \varepsilon_n).$$

Now, we denote by λ_m the extremal distance between K_o and $\bar{G}_{M_m}$, and by λ'_n the extremal distance between $\bar{G}_{M_n}$ and $R - G_{\frac{1}{2}M_n}$ for $m > n$. Then, $\lambda_m \geqq \sum_{n=1}^{m} \lambda'_n$ and $\lim_{m\to\infty} \lambda_m \geqq \sum_{n=1}^{\infty} \lambda'_n$. Since

$$\rho |dz| = \begin{cases} \frac{1}{\sqrt{D_n}} |\operatorname{grad} p_of(z)| \, |dz| & \text{in } G_{M_n} - G_{\frac{1}{2}M_n} \\ 0 & \text{elsewhere} \end{cases}$$

is admissible for the problem of λ_n', we have

$$\lambda_n' \geqq \frac{1}{D_n}\left(\log\frac{1}{\frac{1}{2}M_n} - \log\frac{1}{M_n}\right)^2 = \frac{(\log 2)^2}{D_n}$$

$$> \frac{1}{4(k+\varepsilon_n)}(\log 2)^2.$$

Therefore,

$$\lim_{m\to\infty}\lambda_m \geqq \sum_{n=1}^{\infty}\frac{(\log 2)^2}{4(k+\varepsilon_n)} = \infty .$$

Thus we complete the proof.

Now we show that Theorem 5.7 is an extension of Theorem 5.1. Let F be a closed set in R such that R - F does not belong to $\mathcal{G}_p$. Then, modifying Constantinescu-Cornea's theorem ([4]) (cf. also [17]), we have the following lemma.

Lemma 5.5. Let p be a point of Δ_1 at which f(z) has a limit q in R'^*_N. Then, there exists an sequence $\{a_n\}$ in F such that $\lim_{n\to\infty} f(a_n) = q$.

We define, for a point p of Δ_1, a cluster set $f_F^{\vee}(p)$ as

$$f_F^{\vee}(p) = \bigcap_n \overline{f(F \cap V_n)} ,$$

where V_n is the 1/n -neighborhood of p and the closure is taken in R'^*_N. If f(z) is a GD mapping it has fine limits quasi-everywhere on Δ. And Theorem 5.7 is written as follows.

Theorem 5.7'. Let f(z) be a GD mapping. And let p be a point of of Δ_1 for which there exists a closed set F such that R - F does not belong to $\mathcal{G}_p$ and $f_F^{\vee}(p)$ is one point q. If q satisfies the condition that

$$\varlimsup_{M\to 0} \frac{\mathcal{D}_M}{M^2} < \infty ,$$

then the set E of such points as p has vanishing inner Kuramochi capacity.

We consider the case that R is the unit disk : $|z| < 1$, R' is the Riemann sphere and f(z) is a meromorphic function in R such that

$$\iint \left(\frac{|f'|}{1 + |f|^2} \right)^2 dxdy < \infty .$$

We assume $q = 0$ and take $(|w| \geq 1)$ as K'_0 . Then, $p(w) = \log \frac{1}{|w|}$ and $\zeta = w$. And we have $M = |w|$ $(=r)$ and

$$\mathcal{D}_M = \iint \left| \frac{dw}{dz} \right|^2 dxdy = A(r).$$

Let γ be the radial at $e^{i\theta}$, then $R - \gamma$ does not belong to $\mathcal{G}_{e^{i\theta}}$. Suppose the contrary is true, and take a neighborhood V of $e^{i\theta}$ which does not contain γ. Then $V \cap (R - \gamma)$ belongs to $\mathcal{G}_{e^{i\theta}}$ and there exists a connected component D of $V \cap (R - \gamma)$ which belongs to $\mathcal{G}_{e^{i\theta}}$. Let D' be the symmetric image of D with respect to the radial γ, then it belongs to $\mathcal{G}_{e^{i\theta}}$. But $D \cap D' = \emptyset$. This is a contradiction.

Thus Theorem 5.7' is reduced to Theorem 5.1, but in our case, the set E is of <u>inner</u> capacity zero.

§ 6 Analytic mappings of type Bl

In this section we study an analytic mapping of type Bl of a Riemann surface R into a Riemann surface R'. An analytic mapping f of R into R' is said to be locally of type Bl at $a' \in R'$ if there exists an open set G' of R' containing the point a', such that every connected component of the inverse image $f^{-1}(G')$ of G' is of class SO_{HB}. If f is locally of type Bl at every point of R', it is said to be of type Bl. Let $n_f(a')$ denote the number of points of the set $f^{-1}(a')$, where multiplicities are counted. Then, according to M.Heins, if f is of type Bl, $n_f(a') = \sup_{b' \in R'} n_f(b')$ except for a set of capacity zero.

Here, we concern with the problem at what points an analytic mapping f would be locally of type Bl. For this problem, we refer to the results of Constantinescu-Cornea [4] and of Doob [5]. First, we consider, following Constantinescu-Cornea, Wiener's compactification R_W^* of a Riemann surface R. Let Γ_W be its harmonic boundary and Γ_f be the sebset of Γ_W at each point of which the analytic maping f has continuous extension with a value in R'^*_W. Then, the set of points at which f is locally of type Bl is $R' - \overline{f(\Gamma_f)}$.

Doob considered Martin compactification R_M^* and defined the _essential closed range of the fine boundary function f'_ of the analytic mapping f as follows. The fine boundary function is the function defined on a set on $\Delta = R_M^* - R$ at each point of which f has a fine limit on R'^*_M (Martin compactification of R') with values of the fine limits. Let B be the domain of the fine boundary function f' of f, and B_o be a subset of B of harmonic measure 0. Consider the closure in R'^*_M of $f'(B - B_o)$. The intersection of all these closures (equal to the intersection of countably many, and therefore attained by a proper choice of B_o) is called the essential closed range of the fine boundary function. Using this notion Doob gets the following result :

the necessary and sufficient condition for f to be locally of type B1 at $a' \in R'$ is that a' is in the closure of $f(R)$ but is not in the essential closed range of the fine boundary function f'.

We consider Kuramochi compactification R_N^* of R, and assume that f is a GD mapping.

Theorem 6.1. Let f be a GD mapping of R into R' and locally of type B1 at $a' \in R'$. Then, the set of R_N^* at each point of which f has the fine limit a' has vanishing inner Kuramochi capacity.

Proof Since f is locally of type B1 at $a' \in R'$, there exists an open set G' containing a' such that every connected of $f^{-1}(G')$ is of class SO_{HB}. Let V_1' and V_2' be neighborhoods of a' such that $\bar{V}_2' \subset \bar{V}_1' \subset G'$. We take a connected component G of $f^{-1}(G')$ and put $G_o = G \cap f^{-1}(\bar{V}_1' - \bar{V}_2')$. There exists a continuous Dirichlet function h' in R' such that $h' = 1$ on $\bar{V}_2'$, $= 0$ on $R' - V_1'$ and harmonic in $V_1' - \bar{V}_2'$. Since f is a GD mapping, the extremal distance of $f^{-1}(\bar{V}_2')$ and $R - f^{-1}(V_1')$ is positive, and there exists a Dirichlet function h_o such that $h_o = 1$ on $f^{-1}(\bar{V}_2')$ and $= 0$ in $R - f^{-1}(V_1')$. Since the restriction f_G of f in G is of type B1 as a mapping of G into G', $n_{f_G}(p') = n_{f_G} =$ constant for all points of G' except for points belonging to a set of capacity 0. And, since $h' \circ f$ is a bounded harmonic function (with countable removable singularities) in G_o and G_o belongs to class SO_{HB}, we have

$$h' \circ f = H^{G_o}_{h' \circ f} = H^{G_o}_{h_o} = h_o^{\partial G_o}$$

(for these equalities, cf. [4]). Hence,

$$\|d(h'\circ f)\|_G = \|d(h'\circ f)\|_{G_o} = \|dh_o^{\partial G_o}\|_{G_o} = \|dh_o^{\partial G_o}\|_G.$$

Therefore, we have

$$n_{f_G} \|dh'\|^2_{G'} = \|d(h'\circ f)\|^2_G = \|dh_o^{\partial G_o}\|^2_G$$

and

$$n_{f_G} = \|dh_o^{\partial G_o}\|^2_G \Big/ \|dh'\|^2_G$$

Summing n_{f_G} over all connected components of $f^{-1}(G')$, we have

$$\sum_G n_{f_G} = \sum_G \|dh_o^{\partial G_o}\|^2_G \Big/ \|dh'\|^2_{G'} < \|dh_o\|^2_R \Big/ \|dh'\|^2_{G'} < \infty .$$

Letting V_2' shrink to the point a',

$$\lim_{V_2'\to a'} \sum_G \|dh_o^{\partial G_o}\|^2_G \Big/ \|dh'\|^2_{G'} = \sum_G n_{f_G} < \infty .$$

But when V_2' shrinks to a' $\|dh'\|^2_G$ tends to 0, so $\sum_G \|dh_o^{\partial G_o}\|^2_G$ must tend to 0 That is, the extremal distance between $(R-)f^{-1}(V_1')$ and $f^{-1}(\bar{V}_2')$ will be infinite when V_2' shrinks to the point a'. Therefore, capacity of the set E at each point of which f has the fine limit a' is zero, because if a point a belongs to E, a belongs to $\Delta_1(f^{-1}(V_2'))$.

Next we assume that the GD mapping f is of type Bl on R'. Then, according to Constantinescu-Cornea, if R is hyperbolic R' must also be hyperbolic. And, as is seen in the proof of the previous theorem, f is finitely sheeted. We compactify R' in the sense of Kuramochi, and denote by Δ_N and Δ'_N Kuramochi boundaries of R and R', respectively. Kuramochi called a boundary point <u>singular</u> when it has positive capacity.

Theorem 6.2. A singular boundary point is mapped on a singular boundary point by a GD mapping of type Bl. Especially, if R' is of finite genus, there is no singular point on Δ_n.

Proof Let p be a singular point of Δ_N. Since $\{p\}$ has the positive capacity, the GD mapping f has a fine limit q on R'^*_N at p. We suppose the capacity of $\{q\}$ is zero. Let V_n be a neighborhood of q and ω_n be the equilibrium potential of V_n. Then $D(\omega_n)$ and $D(\omega_n \circ f(z)) = n_f D(\omega_n)$ decrease to zero when n tends to infinity. and since $p \in \bigcap_n \Delta_1 f^{-1}(V_n)$ the capacity of $\{p\}$ must be zero. This is contradiction.

If R' is of finite genus, there is no singular points on Δ'_N. So, R can not have a singular boundary point.

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